Lesson 25 Maclaurin Series

I. Power Series for functions that are not the sum of a geometric series

II. Examples

Quiz Wed (11/1) on Lessons 23 & 24 Geometric Series/ Power Series

* Need calculator

I. Power series for functions that are not the sum of a geometric series

i.e. We cannot make the function look like

$$\left(\frac{x^5}{c}\right)\frac{a}{(1-(kx^t)^n)}$$

T The power series representation for the function f (centered at 0, Maclaurin Series)

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad |x| < R$$

"infinite polynomial form"

radius of convergence. Tells me where f = infinite polynomial

$$= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

has coefficients

$$c_n = \frac{f^{(n)}(0)}{n!}$$

n^{th} derivative of f.

(n) is NOT a power

Rcl: $0! = 1$, $1! = 1$, $2! = 2\cdot 1 = 2$, $3! = 3\cdot 2\cdot 1 = 6$, etc.

Maclaurin series for $f(x)$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

This is really = if $|x| < R$

$$= f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \dots$$

$\boxed{\text{Ex}}$ Find the Maclaurin series for

$$f(x) = \cos(x)$$

$$\begin{aligned}
f(x) &= \cos(x) &&= f^{(4)}(x) &&= f^{(8)}(x)\\
f'(x) &= -\sin(x) &&= f^{(5)}(x) && \vdots\\
f''(x) &= -\cos(x) &&= f^{(6)}(x) && \vdots\\
f'''(x) &= \sin(x) &&= f^{(7)}(x) &&
\end{aligned}$$

$f^{(8)}(0) =$

$$\begin{aligned}
f^{(4)}(0) &= f(0) = \cos(0) = 1\\
f^{(5)}(0) &= f'(0) = -\sin(0) = 0\\
f^{(6)}(0) &= f''(0) = -\cos(0) = -1\\
f^{(7)}(0) &= f'''(0) = \sin(0) = 0
\end{aligned}$$

Mac series for $\cos(x)$

$$1 + \frac{0}{1!}x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5 - \frac{1}{6!}x^6 + \frac{0}{7!}x^7 + \frac{1}{8!}x^8 + \dots$$

$n=0$ $n=1$ $n=2$ $n=3$ $n=4$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$\boxed{Ex}$ Find Maclaurin series for $f(x) = \frac{1}{1-x}$

(If Mac. Series thm works, then this answer should be $\sum_{n=0}^{\infty} 1 \cdot x^n$ $\quad a = 1$, $r = x$)

$f(0) = 0!$ $\quad f(x) = \frac{1}{1-x} = (1-x)^{-1} = 0!\,(1-x)^{-1}$

$f'(0) = 1!$ $\quad f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2} = 1!\,(1-x)^{-2}$

$f''(0) = 2!$ $\quad f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3} = 2!\,(1-x)^{-3}$

$f'''(0) = 3!$ $\quad f'''(x) = -3(2)(1-x)^{-4}(-1) = 3 \cdot 2(1-x)^{-4} = 3!\,(1-x)^{-4}$

$f^{(4)}(0) = 4!$ $\quad f^{(4)}(x) = -4(3)(2)(1-x)^{-5}(-1) = 4 \cdot 3 \cdot 2(1-x)^{-5} = 4!\,(1-x)^{-5}$

$f^{(7)}(x) = 7!\,(1-x)^{-8}$

$f^{(n)}(0) = n!$

Mac Series for $\frac{1}{1-x}$

$$\frac{1}{1-x} = \underbrace{\frac{0!}{1}} + \frac{1!}{1!}x + \frac{2!}{2!}x^2 + \frac{3!}{3!}x^3 + \frac{4!}{4!}x^4 + \cdots$$

if $|x| < 1$

$$= 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$= \sum_{n=0}^{\infty} x^n$$

The following information will be provided to you on Exam 3:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \quad \text{if } |x| < 1$$

Radius of convergence is ∞. These 3 are equal for any x

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad \text{if } |x| < 1$$

II. Examples

$\boxed{Ex}$ Find a Maclaurin series for

a) $f(x) = 5\sin(x^3)$

Formula Sheet $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

$$f(x) = 5\sin(x^3) = 5 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x^3)^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{5(-1)^n}{(2n+1)!} x^{6n+3}$$

$\boxed{Ex}$ Use the first 4 nonzero terms of the Maclaurin Series to approximate the value.

a) $e^{(0.4)}$

1st 4 terms of e^x series

$$e^x \approx 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$e^{(0.4)} \approx 1 + \frac{(0.4)}{1} + \frac{(0.4)^2}{2} + \frac{(0.4)^3}{6} \approx 1.49067$$

(Calculator says $e^{0.4} \approx 1.49182$)

(b) $\ln(0.8)$

1st 4 terms of $\ln(1+x)$ series

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$$

$\ln(0.8)$ using $\ln(1+x)$

$0.8 = 1+x$

$-0.2 = x$

$$\ln(0.8) = \ln(1+(-0.2))$$

$$\approx (-0.2) - \frac{(-0.2)^2}{2} + \frac{(-0.2)^3}{3} - \frac{(-0.2)^4}{4}$$

$$\approx -0.22307$$

(Calculator gives $\ln(0.8) \approx -0.22314$)

$\boxed{Ex}$ Use the first 3 nonzero terms of the series to approximate the integral.

$$\int_0^{0.2} \cos(x^3)\,dx$$

Formula sheet: $\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} = 1 - \frac{x^2}{2} + \frac{x^4}{24}$

$$\Rightarrow \quad \cos(x^3) \approx 1 - \frac{(x^3)^2}{2} + \frac{(x^3)^4}{24}$$

$$= 1 - \frac{1}{2}x^6 + \frac{1}{24}x^{12}$$

$$\int_0^{0.2} \cos(x^3)\,dx \approx \int_0^{0.2} \left(1 - \frac{1}{2}x^6 + \frac{1}{24}x^{12}\right)dx$$

$$= x - \frac{1}{2}\frac{x^7}{7} + \frac{1}{24}\frac{x^{13}}{13} \Big|_0^{0.2}$$

$$= (0.2) - \frac{1}{2}\frac{(0.2)^7}{7} + \frac{1}{24}\frac{(0.2)^{13}}{13} - (0 - 0 + 0)$$

Round to 5 digits

$$\approx 0.19999|9908$$

$$\approx 0.2$$